Journal of Statistical Physics, Vol. 34, Nos. 5/6, 1984

Infinite Conformal Symmetry of Critical Fluctuations in Two Dimensions

A. A. Belavin,[1] A. M. Polyakov,[1] and A. B. Zamolodchikov[1,2]

Received October 27, 1983

We study the massless quantum field theories describing the critical points in two dimensional statistical systems. These theories are invariant with respect to the infinite dimensional group of conformal (analytic) transformations. It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of the Virasoro algebra. Exactly solvable theories associated with degenerate representations are analized. In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the system of linear differential equations.

KEY WORDS: Second order phase transitions; two-dimensional systems; operator algebra; conformal symmetry; Vivasoro algebra; Kac formula.

According to the scaling hypothesis, fluctuations of order parameters right at the point of a second-order phase transition possess invariance under the scaling transformations

$$\xi^a \to \lambda \xi^a \tag{1}$$

where ξ^a are the coordinates; $a = 1, 2, \ldots, \mathscr{D}$. In quantum field theory, taken as a mathematical tool for the theory of second-order phase transitions, the invariance is equivalent to the vanishing of the trace of the stress-energy tensor

$$T_a^a(\xi) = 0 \tag{2}$$

[1] Landau Institute for Theoretical Physics, Academy of Sciences of the USSR, Moscow USSR.

[2] Professor A. B. Zamolodchikov was unable to attend the conference to present this invited paper personally.

0022-4715/84/0300-0763$03.50/0

As is known, under the condition (2) the theory possesses also the invariance under all the coordinate transformations

$$\xi^a \to \eta^a(\xi) \tag{3}$$

having the property that the metric tensor g_{ab} transforms as follows:

$$g_{ab} \to \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'} = \rho(\xi) g_{ab} \tag{4}$$

where ρ is some function of coordinates. The coordinate transformations with this property form the *conformal group*.

The conformal group possesses distinct properties for the cases $\mathscr{D} > 2$ and $\mathscr{D} = 2$. In the former case this group is a finite-dimensional one. The corresponding symmetry in quantum field theory has been extensively studied during the last decade. In particular it was shown that the relevant local fields (order parameters) possess anomalous scale dimensions d_e. Computation of the spectrum $\{d_e\}$ of the anomalous dimensions is the main problem of conformal quantum field theory because these very quantities determine the singularities of thermodynamic functions in the vicinity of the critical point.

Various approaches for solving this problem have been proposed including Hamiltonian theory, the renormalization group, and the conformal bootstrap program. Let us mention the bootstrap approach based on the hypothesis of the operator algebra, which is the strong version of Wilson's operator product expansion. In this approach one assumes the existence of a complete set of local fields $\{A_j(\xi)\}$ which includes, alongside the field itself, all its coordinate derivatives. These fields form the operator algebra

$$A_j(\xi)A_i(0) = \sum_k C_{ji}^k(\xi)A_k(0) \tag{5}$$

where the functions $C_{ij}^k(\xi)$ are the structure constants of the algebra. Combining the self-consistency conditions of the operator algebra (these are locality plus associativity) with the conformal symmetry one gets the system of equations determining, in principle, the values of anomalous dimensions and the structure constants. However, this system turns out to be too cumbersome to be solved explicitly, the main difficulty being the classification of the enormous set of operators $\{A_j(\xi)\}$.

The situation turns out to be much better in this respect in the case of $\mathscr{D} = 2$. The main reason is that the conformal group in this case is an infinite-dimensional one. It consists of all the analytic transformations

$$z \to \zeta(z), \qquad \bar{z} \to \bar{\zeta}(\bar{z}) \tag{6}$$

of complex coordinates

$$z = \xi^1 + i\xi^2, \qquad \bar{z} = \xi^1 - i\xi^2 \tag{7}$$

All the fields $A_j(\xi)$ forming the operator algebra can be classified according to representations of this infinite conformal group. Each representation is infinite dimensional and constitutes an essential part of the complete totality of the fields $\{A_j\}$. So, the structure of the set of fields $\{A_j\}$ turns out to be surveyable; in special cases this set consists of a finite number of representations. From the general point of view the conformal theory in two dimensions can be considered as an instructive example of quantum field theory governed by an infinite-dimensional symmetry group.

In this talk we shall describe briefly some properties of conformal quantum field theory in two dimensions. Apart from describing general properties of the operator algebra goverened by the infinite-dimensional conformal group, we shall show that there are infinitely many special cases of conformal field theory—we call them "minimal" theories—having the following remarkable properties: (i) All the anomalous dimensions in these theories are known exactly; and (ii) the correlation functions of local fields, being the solutions of systems of linear differential equations, can also be computed. From the mathematical point of view these theories are associated with the degenerate representations of conformal algebra. The simplest nontrivial example of a minimal theory turns out to describe the critical theory of the two-dimensional Ising model. The other minimal theories are likely to describe critical points of other spin systems with discrete symmetry groups and seem to deserve the most detailed investigation. An exciting result in this direction was obtained by Dotsenko, who found the minimal theory presumably describing the critical point of the Z_3 Potts model.

Let us list the main statements, leaving aside the details and proofs.

(1) Among the fields $A_j(\xi)$ constituting the operator algebra (5), there are the special operators ϕ_n—we call them *ancestor fields*—possessing the following simple conformal transformation law:

$$\phi_n(z,\bar{z}) \to \frac{d\zeta^{\Delta_n}}{dz}\,\frac{d\bar{\zeta}^{\bar{\Delta}_n}}{d\bar{z}}\,\phi_n(\zeta,\bar{\zeta}) \tag{8}$$

where Δ_n and $\bar{\Delta}_n$ are real parameters which we shall call dimensions. Actually, the quantities

$$d_n = \Delta_n + \bar{\Delta}_n, \qquad S_n = \Delta_n - \bar{\Delta}_n \tag{9}$$

are just the scale dimension and spin of the field ϕ_n, respectively. It follows from locality considerations that the spin S_n can take either integer or half-integer values. The identity operator I is an example of an ancestor field. A nontrivial theory includes more than one ancestor field and the index n is introduced to distinguish between them.

(2) The ancestor fields themselves cannot form the complete set of operators constituting the operator algebra (5). In fact, the complete set $\{A_j(\xi)\}$ of local fields consists of subsets $[\phi_n]$, which we call *conformal classes*, each conformal class corresponding to some ancestor field ϕ_n. Apart from the field ϕ_n, the conformal class $[\phi_n]$ includes infinitely many other fields which can be considered in the descendants of the ancestor field ϕ_n. They possess more complicated transformation properties than ancestor fields. In fact, variations of any field belonging to the conformal class $[\phi_n]$ can be expanded linearly in terms of representatives of the same conformal class, i.e., each conformal class forms a representation of the conformal group. In order to describe the structure of a conformal class some properties of the stress-energy tensor are to be mentioned.

(3) Because of the conservation of the stress-energy tensor

$$\partial_a T^{ab}(\xi) = 0 \tag{10}$$

and the zero-trace condition $T_a^a = 0$, the components

$$\begin{aligned} T &= T_{11} - T_{22} + 2iT_{12} \\ \bar{T} &= T_{11} - T_{22} - 2iT_{12} \end{aligned} \tag{11}$$

satisfy equations

$$\partial_{\bar{z}} T(\xi) = 0, \qquad \partial_z \bar{T}(\xi) = 0 \tag{12}$$

This means that any correlation function of the form

$$\langle T(z) A_{j_1}(\xi_1) \dots A_{j_N}(\xi_N) \rangle \tag{13}$$

is an analytic function of z and possesses poles at the points $z_1, z_2, \dots, z_N$, the orders of the poles and the residues depending on the conformal properties of the fields $A_{j_k}(\xi_k)$. In particular, the following relationship holds:

$$\begin{aligned} &\langle T(z)\phi_1(\xi_1) \dots \phi_N(\xi_N) \rangle \\ &\quad = \sum_{i=1}^{N} \left[\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right] \langle \phi_1(\xi_1) \dots \phi_N(\xi_n) \rangle \end{aligned} \tag{14}$$

where $\phi_1, \dots, \phi_N$ are arbitrary ancestor fields and $\Delta_1, \dots, \Delta_N$ are their dimensions. Actually, the fields $T(z)$ and $\bar{T}(\bar{z})$ represent generators of the conformal group in the field theory. The exact meaning of this statement is given by the formula

$$\delta_\epsilon A_j(z, \bar{z}) = \oint d\zeta\, \epsilon(\zeta) T(\zeta) A_j(z, \bar{z}) \tag{15}$$

where $\delta_\epsilon A_j$ is the variation of the field A_j under the infinitesimal conformal transformation

$$z \to z + \epsilon(z) \tag{16}$$

The integration contour in (15) encircles the point z.

The conformal properties of fields T and $\overline{T}$ themselves are described by the formulas

$$\begin{aligned} \delta_\epsilon T(z) &= \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{C}{12}\epsilon'''(z) \\ \delta_\epsilon \overline{T}(\bar{z}) &= 0 \end{aligned} \tag{17}$$

where the prime denotes the derivative $\partial/\partial z$. The real positive constant C is a parameter of the theory. It is convenient for later use to introduce the set of operators

$$\left.\begin{aligned} L_n(z) &= \oint_z T(\zeta)(\zeta - z)^{n+1}\, d\zeta \\ \overline{L}_n(\bar{z}) &= \oint_{\bar{z}} \overline{T}(\bar{\zeta})(\bar{\zeta} - \bar{z})^{n+1}\, d\bar{\zeta} \end{aligned}\right\} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{18}$$

where the contours encircle the points z and $\bar{z}$. These operators satisfy the commutation relations

$$\begin{aligned} \left[L_n(z), L_m(z) \right] &= (n - m)L_{n+m}(z) + \frac{C}{12}(n^3 - n)\delta_{n+m,0} \\ \left[L_n(z), \overline{L}_m(\bar{z}) \right] &= 0 \end{aligned} \tag{19}$$

which are equivalent to the formulas (17). These relations are well known in dual theory as the *Virasoro algebra*: the constant C is known as the *central charge*. It is worth mentioning that the dual theories can be considered as special cases of conformal quantum field theory in two dimensions.

(4) It can easily be shown that any ancestor field ϕ_n satisfies the equations

$$\begin{aligned} L_\kappa(z)\phi_n(z,\bar{z}) &= 0 \qquad \text{for} \quad \kappa > 0 \\ L_0(z)\phi_n(z,\bar{z}) &= \Delta_n \phi_n(z,\bar{z}) \end{aligned} \tag{20}$$

In the general case the conformal class $[\phi_n]$ consists of all the fields of the form

$$\begin{aligned} &\phi_n^{\{-\kappa_1, \ldots, -\kappa_N\}\{-\bar{\kappa}_1, \ldots, -\bar{\kappa}_M\}}(z,\bar{z}) \\ &\qquad = \left[L_{-\kappa_1}(z) \ldots L_{-\kappa_N}(z)\overline{L}_{-\bar{\kappa}_1}(\bar{z}) \ldots \overline{L}_{-\bar{\kappa}_M}(\bar{z})\phi_n(z,\bar{z}) \right] \end{aligned} \tag{21}$$

with arbitrary integers $\kappa, \bar{\kappa}, N, M$. The dimensions of the descendants

$\phi_n^{\{\kappa\}\{\bar\kappa\}}$ are equal to

$$\Delta_n^{\{\kappa\}} = \Delta_n + \sum_{i=1}^{N} \kappa_i, \qquad \bar\Delta_n^{\{\bar\kappa\}} = \bar\Delta_n + \sum_{i=1}^{N} \bar\kappa_i \tag{22}$$

and differ by integers from the dimensions $\Delta_n, \bar\Delta_n$ of the corresponding ancestor field ϕ_n.

It is essential that correlation functions of the descendants can be expressed in terms of correlation functions of the corresponding ancestor fields by means of linear differential operators. For example,

$$\begin{aligned}&\left\langle \phi_n^{(-\kappa_1, -\kappa_2, \dots, -\kappa_M)}(z)\phi_1(\xi_1) \dots \phi_N(\xi_N)\right\rangle \\ &\quad = \hat{\mathscr{L}}_{-\kappa_M}(z, z_i) \dots \hat{\mathscr{L}}_{-\kappa_1}(z, z_i)\langle \phi_n(z)\phi_1(\xi_1) \dots \phi_N(\xi_N)\rangle\end{aligned} \tag{23}$$

where

$$\hat{\mathscr{L}}_{-\kappa}(z, z_i) = \sum_{i=1}^{N} \frac{(1-\kappa)\Delta_i}{(z-z_i)^{\kappa}} - \frac{1}{(z-z_i)^{\kappa}} \frac{\partial}{\partial z_i} \tag{24}$$

Therefore, all the information about conformal field theory is concentrated in the correlation functions of the ancestor fields ϕ_n.

(5) The operator expansion of two ancestor fields can be represented in the following symbolic form

$$\phi_n(\xi)\phi_m(0) = \sum_p C_{nm}^p[\phi_p] \tag{25}$$

where C_{nm}^p are the coefficients standing in the right-hand side in front of the ancestor fields $\phi_p(0)$ and the brackets represent the contribution of all the descendants of the field ϕ_p. The coefficients standing in front of these descendants are determined uniquely by the conformal invariance of the expansion (25). Therefore, the associativity condition for the operator algebra (5), which is equivalent to the crossing-symmetry condition for the four-point functions

$$\langle \phi_n(\xi_1)\phi_m(\xi_2)\phi_\kappa(\xi_3)\phi_l(\xi_4)\rangle \tag{26}$$

leads to the system of equations for the coefficients C_{nm}^p and the dimensions Δ_n of ancestor fields only. We have not yet investigated this system in the general case, but there exist particular solutions which are associated with the degenerate representations of the conformal group; these will be described below.

(6) Let us consider some ancestor field ψ_Δ. Under appropriate choice of Δ, it is possible that the corresponding conformal class $[\psi_\Delta]$ contains less fields than usual. This happens when some of the descendants $\chi_{\Delta+m}$ of ψ_Δ, formally computed according to (21), turn out to possess themselves the

conformal properties (8) of the ancestor field

$$\chi_{\Delta+M}(z) \to \left(\frac{d\zeta}{dz} \right)^{\Delta+M} \chi_{\Delta+M}(\zeta) \tag{27}$$

(here we neglect the $\bar{z}$ dependence of the fields). We shall refer to such superfluous ancestor operators as *null fields*. To obtain the true irreducible conformal class $[\psi_\Delta]$ in this situation one should put this null-field equal to zero.

$$\chi_{\Delta+M} = 0 \tag{28}$$

This also kills all the descendants of the null field

$$[\chi_{\Delta+M}] = 0 \tag{29}$$

So, in such a case, the conformal class $[\psi_\Delta]$ contains less fields than the usual one, and we call it a *degenerate* one; we shall also call the corresponding ancestor field ψ_Δ degenerate.

In the case of general Δ there are no null fields. The null fields appear under appropriate choice of this parameter. For example, if Δ is equal to

$$\Delta = \frac{5 - C \pm \left[(C-1)(C-25) \right]^{1/2}}{16} \tag{30}$$

the operator

$$\chi_{\Delta+2}(\xi) = \phi_\Delta^{(-2)}(\xi) + \frac{3}{2(2\Delta+1)} \frac{\partial^2}{\partial z^2} \phi_\Delta(\xi) \tag{31}$$

is a null field. All the values of Δ which lead to degenerate conformal classes $[\psi_\Delta]$ are known from the theory of representations of the Virasoro algebra. The corresponding formula was discovered by Kac and has the form

$$\Delta = \Delta_{(n,m)} = \Delta_0 + \left(\frac{\alpha_+}{2} n + \frac{\alpha_-}{2} m \right)^2 \tag{32}$$

where

$$\Delta_0 = \frac{C-1}{24}, \qquad \alpha \pm = \frac{(1-C)^{1/2} \pm (25-C)^{1/2}}{\sqrt{24}} \tag{33}$$

while n, m are arbitrary positive integers. Let us denote by $\psi_{(n,m)}(z)$ the degenerate field having the dimension $\Delta_{(n,m)}$. Note that

$$\Delta_{(1,1)} = 0 \quad \text{and hence} \quad \psi_{(1,1)} = I \tag{34}$$

For the representation $[\psi_{(n,m)}]$ the corresponding null field has the dimension

$$\Delta_{(n,m)} + n \cdot m \tag{35}$$

Note also that the values $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ coincide with the two values given by (30).

The degenerate fields possess a remarkable property. The correlation functions which include the degenerate fields

$$\langle \psi_{(n,m)}(z)\phi_1(\xi_1) \ldots \phi_N(\xi_N)\rangle \tag{36}$$

satisfy linear differential equations. The reason is simple. Since correlators of *any* descendants are expressed in terms of the correlators of ancestor fields in terms of linear differential operators, the same is true for the null fields. Taking into account (28) one gets the differential equations. For example, the correlation functions of the fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$ satisfy the second-order differential equation, which follows from (31),

$$\left\{ \frac{3}{2(2\Delta_{(1,2)}+1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N} \left[\frac{\Delta_i}{(z-z_i)^2} - \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right] \right\}$$
$$\times \langle \psi_{(1,2)}(z)\phi_1(z_1) \ldots \phi_N(z_N)\rangle = 0 \tag{37}$$

In the general case, the correlation functions of the fields $\psi_{(n,m)}(z)$ satisfy linear differential equations of order $n \cdot m$.

(7) It can be shown, using the differential equations, that the degenerate fields form a closed operator algebra. More precisely, the conformal classes $[\psi_{(n,m)}]$ of the degenerate fields constitute the closed algebra. The qualitative structure of this algebra is described by the following symbolic formula:

$$\psi_{(n_1,m_1)}\psi_{(n_2,m_2)} = \sum_{\kappa=|n_1-n_2|+1}^{n_1+n_2+1} \sum_{l=|m_1-m_2|+1}^{m_1+m_2+1} \left[\psi_{(\kappa,l)}\right] \tag{38}$$

where the brackets represent the contributions of conformal classes of the corresponding degenerate fields $\psi_{(\kappa,l)}$ and the coefficients in the right-hand side are suppressed. Here the index κ runs over even values if the sum $n_1 + n_2$ is odd and vice versa; the same is true for the index l.

The closed algebra of degenerate fields gives rise to the idea of considering it as the operator algebra of a certain conformal quantum field theory. To examine this idea let us concentrate again on the Kac formula (32). One can easily note that there are three different domains of the parameter C. If $C > 25$ the dimensions $\Delta_{(n,m)}$ become negative for sufficiently large n and m; if $1 < C < 25$ the dimensions $\Delta_{(n,m)}$ take, in general, complex values. Neither possibility is very attractive. Therefore we shall

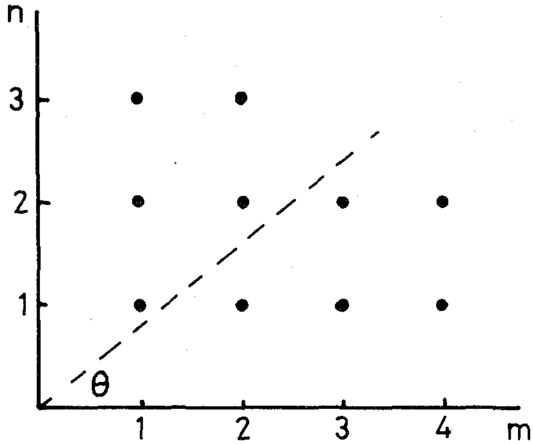

Fig. 1. Diagram of dimensions with physical values of n and m marked by dots; the slope $\tan\theta$ of the dotted line is as in Eq. (40).

concentrate on the domain

$$0 < C < 1 \tag{39}$$

To understand the situation taking place here it is useful to consider the "diagram of dimensions" shown in Fig. 1. This is the plane with coordinates n, m. The "physical," i.e., positive integer values of n and m, are marked by dots. The dotted line has the slope

$$\tan\theta = -\frac{\alpha+}{\alpha-} = \frac{(25-C)^{1/2}-(1-C)^{1/2}}{(25-C)+(1-C)} \tag{40}$$

Each point of the plane corresponds to the value

$$\Delta = \Delta_0 + \alpha^2$$

of the dimension, where α is proportional to the distance between the point and the dotted line. If $C < 1$ then

$$\Delta_0 < 0 \tag{41}$$

and one can see from Fig. 1 that at arbitrary values of the slope $\tan\theta$ we meet again the problem of negative dimensions.

It can be shown, however, that if the slope $\tan\theta$ takes a rational value, i.e.,

$$\frac{(25-C)^{1/2}-(1-C)^{1/2}}{(25-C)^{1/2}+(1-C)^{1/2}} = \frac{p}{q} \tag{42}$$

where p and q are positive integers, then the degenerate conformal classes $[\psi_{(n,m)}]$ with

$$0 < n < p, \qquad 0 < m < q \tag{43}$$

form a closed algebra themselves. This algebra contains a finite number, $(p-1)(q-1)/2$, of conformal classes. Due to the restriction $0 < C < 1$, the integers p and q should satisfy inequalities

$$q > p \quad \text{and} \quad 3p > 2q \tag{44}$$

These operator algebras do not include fields of negative dimension and can be interpreted as field theories. They are just the minimal theories we have mentioned above.

The most simple choice of p and q satisfying (44) is

$$p/q = 3/4 \tag{45}$$

which corresponds to

$$C = 1/2 \tag{46}$$

The associated diagram of dimensions is shown in Fig. 2. Let us list the values of dimensions corresponding to the dots in this figure

$$\begin{aligned} \Delta_{(1,1)} &= \Delta_{(2,3)} = 0 \\ \Delta_{(2,1)} &= \Delta_{(1,3)} = 1/2 \\ \Delta_{(1,2)} &= \Delta_{(2,2)} = 1/16 \end{aligned} \tag{47}$$

In this case we can deal with three degenerate ancestor fields

$$\begin{aligned} I &= \psi_{(1,1)} = \psi_{(2,3)} \\ \epsilon &= \psi_{(2,1)} = \psi_{(1,3)} \\ \sigma &= \psi_{(1,2)} = \psi_{(2,2)} \end{aligned} \tag{48}$$

The qualitative features of the corresponding operator algebra are described by the symbolic formulas

$$\begin{aligned} &[I][I] = [I], \qquad [\epsilon][\epsilon] = [I] \\ &[I][\sigma] = [\sigma], \qquad [\epsilon][\sigma] = [\sigma] \\ &[I][\epsilon] = [\epsilon], \qquad [\sigma][\sigma] = [I] + [\epsilon] \end{aligned} \tag{49}$$

It can be shown that this minimal model describes the critical theory of the Ising model, the fields σ and ϵ being identified with the spin and energy density respectively.

Let us also present the diagram of dimensions, Fig. 3, for the next minimal theory, characterised by the values

$$p/q = 4/5, \qquad C = 7/10 \tag{50}$$

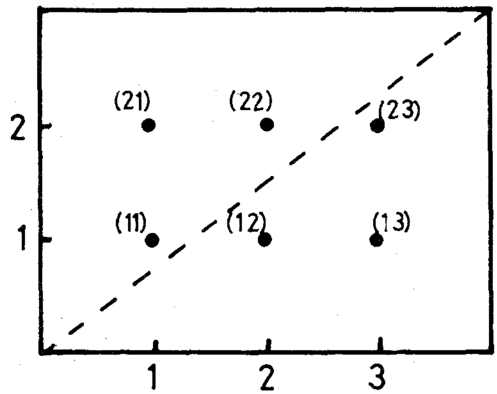

Fig. 2. Diagram of dimensions for $p/q = 3/4$; the dimensions for the dots are given in Eq. (47).

The numerical values of dimensions are

$$\begin{aligned}
\Delta_{(11)} &= \Delta_{(37)} = 0 \\
\Delta_{(12)} &= \Delta_{(33)} = 1/10 \\
\Delta_{(13)} &= \Delta_{(32)} = 3/5 \\
\Delta_{(14)} &= \Delta_{(31)} = 3/2 \\
\Delta_{(22)} &= \Delta_{(23)} = 3/80 \\
\Delta_{(24)} &= \Delta_{(21)} = 7/16
\end{aligned} \tag{51}$$

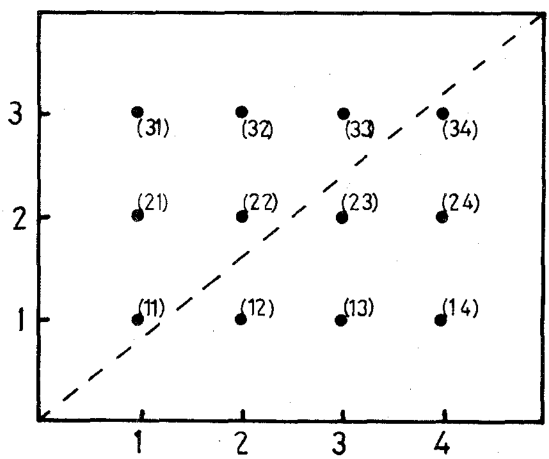

Fig. 3. Diagram of dimensions for $p/q = 4/5$; the numerical values of the dimensions are given in Eq. (51).

The two minimal models mentioned above are the simplest representatives of the series

$$p = q - 1, \qquad q \geqslant 4, \qquad C = 1 - 6/pq \tag{52}$$

The corresponding spectra of dimensions are given by the formula

$$\begin{gathered} \Delta_{(n,m)} = \frac{(pn - qm)^2 - 1}{4pq}, \qquad p = q - 1 \\ 0 < n < p, \qquad 0 < m < q \end{gathered} \tag{53}$$

The minimal theory with

$$p/q = 5/6, \qquad C = 4/5 \tag{54}$$

was investigated by Dotsenko and presumably corresponds to the z_3 Potts model.

In general, every minimal model is a self-consistent conformal quantum field theory and could describe some critical theory. The identification of these theories with the concrete statistical systems basically remains an open problem.

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